"Symmetry, as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection." *Hermann Weyl, Symmetry*

13 SYMMETRIC SPACES

Euclid implicitly assumed that metric relations are unaffected by translations or rotations. Real gravitational fields do not usually have such a high degree of symmetry, but they often admit some group of approximate symmetry transformations, and when they do, we can use this information to help solve the Einstein equations, or even to do without a solution. I shall give only a very brief introduction to the elaborate mathematical theory of symmetric spaces, with special attention to the maximally symmetric spaces that are of special interest in cosmology.

The initial difficulty here is: How can we use some supposed symmetry of a metric space to gain information about the metric, when we need to know the metric before we can establish a coordinate system in which to define the symmetry? In order to avoid this impasse, we shall have to learn ways of describing symmetries in a covariant language, which does not depend on any particular choice of coordinate system. Once this language is established, it becomes a matter of mathematical manipulation to determine those properties of a metric that follow from its symmetries.

1 Killing Vectors

A metric $g_{\mu\nu}(x)$ is said to be *form-invariant* under a given coordinate transformation $x \to x'$, when the transformed metric $g'_{\mu\nu}(x')$ is the same function of its argument x'^{μ} as the original metric $g_{\mu\nu}(x)$ was of *its* argument x^{μ}, that is,

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y) \qquad \text{for all } y \tag{13.1.1}$$

[This is different from the condition for a scalar, which is that $S'(x') = S(x)$.] At any given *point* the transformed metric is given by the relation

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

or equivalently

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x')$$

When (13.1.1) is valid, we can replace $g'_{\rho\sigma}(x')$ with $g_{\rho\sigma}(x')$ and so obtain the fundamental requirement for a form invariance of the metric:

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') \tag{13.1.2}$$

Any transformation $x \to x'$ that satisfies (13.1.2) is called an *isometry*.

In general, Eq. (13.1.2) is a very complicated restriction on the function $x'^\mu(x)$. It can be greatly simplified by descending to the special case of an infinitesimal coordinate transformation:

$$x'^\mu = x^\mu + \varepsilon\xi^\mu(x) \qquad \text{with } |\varepsilon| \ll 1 \tag{13.1.3}$$

To first order in ε, Eq. (13.1.2) now reads

$$0 = \frac{\partial\xi^\mu(x)}{\partial x^\rho} g_{\mu\sigma}(x) + \frac{\partial\xi^\nu(x)}{\partial x^\sigma} g_{\rho\nu}(x) + \xi^\mu(x)\frac{\partial g_{\rho\sigma}(x)}{\partial x^\mu} \tag{13.1.4}$$

This can be rewritten in terms of derivatives of the covariant components $\xi_\sigma \equiv g_{\mu\sigma}\xi^\mu$:

$$\begin{aligned} 0 &= \frac{\partial\xi_\sigma}{\partial x^\rho} + \frac{\partial\xi_\rho}{\partial x^\sigma} + \xi^\mu\left[\frac{\partial g_{\rho\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\mu}}{\partial x^\sigma}\right] \\ &= \frac{\partial\xi_\sigma}{\partial x^\rho} + \frac{\partial\xi_\rho}{\partial x^\sigma} - 2\xi_\mu\Gamma^\mu_{\rho\sigma} \end{aligned}$$

or, more compactly,

$$0 = \xi_{\sigma;\rho} + \xi_{\rho;\sigma} \tag{13.1.5}$$

Any four-vector field $\xi_\sigma(x)$ that satisfies Eq. (13.1.5) will be said to form a *Killing vector*[1] of the metric $g_{\mu\nu}(x)$. The problem of determining all infinitesimal isometries of a given metric is now reduced to the problem of determining all Killing vectors of the metric. Any linear combination of Killing vectors (with constant coefficients) is a Killing vector, so it is the space of vector fields spanned by the Killing vectors that really determines the infinitesimal isometries of a metric.

The Killing condition (13.1.5) is much more restrictive than it looks, for it allows us to determine the whole function $\xi_\mu(x)$ from given values of ξ_σ and $\xi_{\sigma;\rho}$

at some point X. To see this, we need only recall our formula (6.5.1) for the commutator of two covariant derivatives,

$$\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} = -R^{\lambda}_{\sigma\rho\mu}\xi_{\lambda} \tag{13.1.6}$$

and the cyclic sum rule (6.6.5) for the curvature tensor,

$$R^{\lambda}_{\sigma\rho\mu} + R^{\lambda}_{\mu\sigma\rho} + R^{\lambda}_{\rho\mu\sigma} = 0 \tag{13.1.7}$$

By adding (13.1.6) and its two cyclic permutations, we find that any vector ξ_{μ} must satisfy the relation

$$0 = \xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} + \xi_{\mu;\sigma;\rho} - \xi_{\mu;\rho;\sigma} + \xi_{\rho;\mu;\sigma} - \xi_{\rho;\sigma;\mu} \tag{13.1.8}$$

For a Killing vector, (13.1.5) and (13.1.8) give

$$0 = \xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} - \xi_{\mu;\rho;\sigma}$$

and thus (13.1.6) becomes

$$\xi_{\mu;\rho;\sigma} = -R^{\lambda}_{\sigma\rho\mu}\xi_{\lambda} \tag{13.1.9}$$

Hence, given ξ_{λ} and $\xi_{\lambda;\nu}$ at some point X, we can determine the second derivatives of $\xi_{\lambda}(x)$ at X from Eq. (13.1.9), and we can find successively higher derivatives of ξ_{λ} at X by taking derivatives of Eq. (13.1.9). All the derivatives of ξ_{λ} at X will thus be determined as linear combinations of $\xi_{\lambda}(X)$ and $\xi_{\lambda;\nu}(X)$. The function $\xi_{\lambda}(x)$ can then (when it exists) be constructed as a Taylor series in $x^{\lambda} - X^{\lambda}$ within some finite neighborhood of X, and will again be linear in the initial values $\xi_{\lambda}(X)$, $\xi_{\lambda;\nu}(X)$. Thus any particular Killing vector $\xi_{\rho}{}^{n}(x)$ of the metric $g_{\mu\nu}(x)$ can be expressed as

$$\xi_{\rho}{}^{n}(x) = A_{\rho}{}^{\lambda}(x; X)\xi_{\lambda}{}^{n}(X) + B_{\rho}{}^{\lambda\nu}(x; X)\xi_{\lambda;\nu}{}^{n}(X) \tag{13.1.10}$$

where $A_{\rho}{}^{\lambda}$ and $B_{\rho}{}^{\lambda\nu}$ are functions that of course depend on the metric and on X, but do not depend on the initial values $\xi_{\lambda}(X)$ and $\xi_{\lambda;\nu}(X)$, and hence are the same for all Killing vectors. Each Killing vector $\xi_{\rho}(x)$ of a given metric is uniquely specified by the values of $\xi_{\rho}(X)$ and $\xi_{\rho;\sigma}(X)$ at any particular point X.

A set of Killing vectors $\xi_{\rho}{}^{n}(x)$ is said to be *independent* if they do not satisfy any linear relations of the form

$$\sum_{n} c_{n}\xi_{\rho}{}^{n}(x) = 0 \tag{13.1.11}$$

with *constant* coefficients c_{n}. Equation (13.1.10) tells us that *there can be at most* $N(N + 1)/2$ *independent Killing vectors in* N *dimensions*. For consider any M Killing vectors $\xi_{\rho}{}^{n}(x)$. For each n, there are N quantities $\xi_{\rho}{}^{n}(X)$ and $N(N - 1)/2$ independent quantities $\xi^{n}_{\rho;\nu}(X)$ [recall Eq. (13.1.5)], so we can think of the quantities $\xi_{\rho}{}^{n}(X)$ and $\xi^{n}_{\rho;\nu}(X)$ as the components of M vectors in an $N(N + 1)/2$

dimensional space. If $M > N(N+1)/2$, then these M vectors cannot be linearly independent, so they must satisfy relations of the form

$$\sum_n c_n \xi_\rho{}^n(X) = \sum_n c_n \xi^n_{\rho;\nu}(X) = 0$$

Equation (13.1.10) then tells us that the Killing vectors $\xi_\rho{}^n(x)$ satisfy the relations (13.1.11) everywhere, and are therefore not independent Killing vectors.

This result is significant only because we defined independent Killing vectors as vectors that are not subject to any linear relations with *constant* coefficients. At some given point X in an N-dimensional space, any set of more than N Killing vectors will of course be subject to one or more linear relations such as (13.1.11). However, the coefficients c_n in these linear relations need not be constant in x^μ. The above theorem says that any set of more than $N(N+1)/2$ Killing vectors will be subject to linear relations with constant coefficients.

A metric space is said to be *homogeneous* if there exist infinitesimal isometries (13.1.3) that carry any given point X into any other point in its immediate neighborhood. That is, the metric must admit Killing vectors that at any given point take all possible values. In particular, in an N-dimensional space we can choose a set of N Killing vectors $\xi_\lambda{}^{(\mu)}(x; X)$ with

$$\xi_\lambda{}^{(\mu)}(X; X) = \delta_\lambda{}^\mu$$

These are evidently independent, because any relation of the form $c_\mu \xi_\nu{}^{(\mu)}(x; X) = 0$ would at $x = X$ imply that all c_λ vanish.

A metric space is said to be *isotropic* about a given point X if there exist infinitesimal isometries (13.1.3) that leave the point X fixed, so that $\xi^\lambda(X) = 0$, and for which the first derivatives $\xi_{\lambda;\nu}(X)$ take all possible values, subject only to the antisymmetry condition (13.1.5). In particular, in N dimensions we can choose a set of $N(N-1)/2$ Killing vectors $\xi_\lambda^{(\mu\nu)}(x; X)$ with

$$\begin{aligned}
\xi_\lambda{}^{(\mu\nu)}(x; X) &\equiv -\xi_\lambda{}^{(\nu\mu)}(x; X) \\
\xi_\lambda{}^{(\mu\nu)}(X; X) &\equiv 0 \\
\xi_{\lambda;\rho}{}^{(\mu\nu)}(X; X) &\equiv \left[\frac{\partial}{\partial x^\rho}\, \xi_\lambda{}^{(\mu\nu)}(x; X)\right]_{x=X} \equiv \delta_\lambda{}^\mu \delta_\rho{}^\nu - \delta_\rho{}^\mu \delta_\lambda{}^\nu
\end{aligned}$$

These are independent, because any relation of the form $c_{\mu\nu}\xi_\lambda^{(\mu\nu)}(x; X) = 0$ with $c_{\mu\nu} = -c_{\nu\mu}$ would at X imply that $c_{\lambda\rho} - c_{\rho\lambda} = 2c_{\lambda\rho} = 0$.

We shall also have to deal with spaces that are isotropic about *every* point. In this case there are Killing vectors $\xi_\lambda^{(\mu\nu)}(x; X)$ and $\xi_\lambda^{(\mu\nu)}(x; X + dX)$ that satisfy the above initial conditions at X and at $X + dX$, respectively. Any linear combination of these will be a Killing vector, and so $\partial\xi_\lambda^{(\mu\nu)}(x; X)/\partial X^\rho$ will also be a Killing vector of the metric. In order to evaluate this Killing vector at $x = X$ we need only recall that $\xi_\lambda^{(\mu\nu)}(X; X)$ vanishes, and therefore

$$0 = \frac{\partial}{\partial X^\rho}\, \xi_\lambda{}^{(\mu\nu)}(X; X) = \left[\frac{\partial \xi_\lambda{}^{(\mu\nu)}(x; X)}{\partial x^\rho}\right]_{x=X} + \left[\frac{\partial \xi_\lambda{}^{(\mu\nu)}(x; X)}{\partial X^\rho}\right]_{x=X}$$

This gives

$$\left[\frac{\partial}{\partial X^\rho}\,\xi_\lambda{}^{(\mu\nu)}(x;\,X)\right]_{x=X} = -\delta_\lambda{}^\mu\delta_\rho{}^\nu + \delta_\rho{}^\mu\delta_\lambda{}^\nu$$

It is now obvious that we can construct a Killing vector $\xi_\lambda(x)$ that takes any arbitrary value a_λ at $x = X$; we need only take

$$\xi_\lambda(x) = \frac{a_\nu}{N-1}\,\frac{\partial}{\partial X^\rho}\,\xi_\lambda^{(\rho\nu)}(x;\,X)$$

Hence *any space that is isotropic about every point is also homogeneous.*

A metric that admits the maximum number $N(N+1)/2$ of Killing vectors is said to be *maximally symmetric.* In particular, a space that is both homogeneous and isotropic about some given point X will admit the $N(N+1)/2$ Killing vectors $\xi_\lambda^{(\mu)}(x;\,X)$ and $\xi_\lambda^{(\mu\nu)}(x;\,X)$. These Killing vectors are obviously independent, for if they satisfy a linear relation

$$0 = c_\mu\xi_\lambda{}^{(\mu)}(x;\,X) + c_{\mu\nu}\xi_\lambda{}^{(\mu\nu)}(x;\,X)$$

$$c_{\mu\nu} = -c_{\nu\mu}$$

then differentiating with respect to x^ρ and setting $x = X$ gives $c_{\lambda\rho} = 0$, and setting $x = X$ then gives $c_\lambda = 0$. Thus *a homogeneous space that is isotropic about some point is maximally symmetric.* It then also follows that *any space that is isotropic about every point is maximally symmetric.*

We can also prove the converse, that *a maximally symmetric space is necessarily homogeneous and isotropic about all points.* If there are $N(N+1)/2$ independent Killing vectors $\xi_\lambda{}^n(x)$, then we can think of the quantities $\xi_\rho{}^n(X)$, $\xi_{\lambda;\nu}{}^n(X)$ as forming a square matrix, with $N(N+1)/2$ rows labeled by n, and $N(N+1)/2$ columns labeled by the N values of ρ and the $N(N-1)/2$ values of λ and ν with $\lambda > \nu$. Furthermore, this matrix must have a nonvanishing determinant, because any relation of the form

$$\sum_n c_n\xi_\rho{}^n(X) = \sum_n c_n\xi_{\lambda;\nu}{}^n(X) = 0$$

would with (13.1.10) imply that $\sum_n c_n\xi_\rho{}^n(x)$ vanishes, contrary to our assumption that these Killing vectors are independent. It must therefore be possible, for any "row vector" with "components" a_μ and $b_{\mu\nu} = -b_{\nu\mu}$, to find a solution of the equations

$$\sum_n d_n\xi_\mu{}^n(X) = a_\mu$$

$$\sum_n d_n\xi_{\mu;\nu}{}^n(X) = b_{\mu\nu}$$

Hence we can find a Killing vector $\xi_\mu(x)$ for which $\xi_\mu(X)$ takes the value a_μ and $\xi_{\mu;\nu}(X)$ takes the value $b_{\mu\nu}$, by choosing

$$\xi_\mu(x) = \sum_n d_n\xi_\mu{}^n(x)$$

But a_μ is arbitrary, so the space is homogeneous, and $b_{\mu\nu}$ is arbitrary (except that $b_{\mu\nu} = -b_{\nu\mu}$), so the space is isotropic about X.

As an example of a maximally symmetric space, consider an N-dimensional flat space, with vanishing curvature tensor. We can then choose Cartesian coordinates with a constant metric and vanishing affine connection. In this coordinate system, Eq. (13.1.9) reads

$$\frac{\partial^2 \xi_\mu}{\partial x^\rho \partial x^\sigma} = 0$$

The solution is

$$\xi_\mu(x) = a_\mu + b_{\mu\nu}x^\nu$$

with a_μ and $b_{\mu\nu}$ constant. This satisfies the Killing vector condition (13.1.5) if and only if

$$b_{\mu\nu} = -b_{\nu\mu}$$

We can thus choose a set of $N(N+1)/2$ Killing vectors as follows:

$$\xi_\mu^{(\nu)}(x) \equiv \delta_\mu{}^\nu$$
$$\xi_\mu^{(\nu\lambda)}(x) \equiv \delta_\mu{}^\nu x^\lambda - \delta_\mu{}^\lambda x^\nu$$

and the general Killing vector is

$$\xi_\mu(x) = a_\nu \xi_\mu^{(\nu)}(x) + b_{\nu\lambda}\xi_\mu^{(\nu\lambda)}(x)$$

The N vectors $\xi_\mu^{(\nu)}(x)$ represent translations, whereas the $N(N-1)/2$ vectors $\xi_\mu^{(\nu\lambda)}$ represent infinitesimal rotations (or, for a Minkowski space, Lorentz transformations). Thus any flat metric admits $N(N+1)/2$ independent Killing vectors, and is therefore maximally symmetric.

Of course, not all metrics admit the maximum number of Killing vectors. Whether (13.1.9) is soluble for a given set of initial data $\xi_\lambda(X)$, $\xi_{\lambda;\rho}(X)$ depends on the integrability of this equation, which in turn depends on the metric. One integrability condition we shall use below follows from the general formula for commutators of covariant derivatives of tensors:

$$\xi_{\rho;\mu;\sigma;\nu} - \xi_{\rho;\mu;\nu;\sigma} = -R^\lambda_{\rho\sigma\nu}\xi_{\lambda;\mu} - R^\lambda_{\mu\sigma\nu}\xi_{\rho;\lambda}$$

Equation (13.1.9) will satisfy this condition if and only if

$$R^\lambda_{\sigma\rho\mu}\xi_{\lambda;\nu} - R^\lambda_{\nu\rho\mu}\xi_{\lambda;\sigma} + (R^\lambda_{\sigma\rho\mu;\nu} - R^\lambda_{\nu\rho\mu;\sigma})\xi_\lambda$$
$$= -R^\lambda_{\rho\sigma\nu}\xi_{\lambda;\mu} - R^\lambda_{\mu\sigma\nu}\xi_{\rho;\lambda}$$

or, using (13.1.5),

$$[-R^\lambda_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda_{\sigma\rho\mu}\delta^\kappa_\nu + R^\lambda_{\nu\rho\mu}\delta^\kappa_\sigma]\xi_{\lambda;\kappa} = [R^\lambda_{\sigma\rho\mu;\nu} - R^\lambda_{\nu\rho\mu;\sigma}]\xi_\lambda \quad (13.1.12)$$

These conditions are of course empty for a flat space, but in general they will impose linear relations among the ξ_λ and $\xi_{\lambda;\kappa}$ at any given point. Alternatively, if

we know something about the Killing vectors admitted by an unknown metric, then we can use (13.1.12) to learn something about its curvature tensor. In this way, we shall be able in the following sections to deduce the form of a maximally symmetric metric from its isometries.

It should be emphasized that the existence of a definite number of independent Killing vectors does not depend on a particular choice of coordinate system. If $\xi^\mu(x)$ is a Killing vector of a metric $g_{\mu\nu}(x)$, then by performing a coordinate transformation $x^\mu \to x'^\mu$ we obtain a metric

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

and, since (13.1.5) is generally covariant, this obviously has a Killing vector

$$\xi'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\xi^\nu(x)$$

If M Killing vectors $\xi_\mu{}^n(x)$ are independent, then so are the M Killing vectors $\xi_\mu{}^{n'}(x')$, for any linear relation among the $\xi^{n'}$ would imply a linear relation among the ξ^n. Thus the maximal symmetry of a given space is an inner property, not depending on how we choose the coordinate system. In particular, it follows that any space with vanishing curvature tensor is maximally symmetric; the converse, however, is not true. It is also easy to see that the homogeneity or isotropy of a given space is independent of the choice of coordinates. As far as these simple symmetries are concerned, we have accomplished the task laid out in the introduction to this chapter, that of describing symmetries of the metric in a generally covariant language.

2 Maximally Symmetric Spaces: Uniqueness

We now show that the maximally symmetric spaces are uniquely specified by a "curvature constant" K, and by the numbers of eigenvalues of the metric that are positive or negative. That is, given *two maximally symmetric metrics with the same K and the same numbers of eigenvalues of each sign, it will always be possible to find a coordinate transformation that carries one metric into the other.* Armed with this theorem, we shall be able in the next section to carry out an exhaustive study of maximally symmetric spaces by simply constructing such metrics in one convenient coordinate system.

We showed in the last section that at any given point x in a maximally symmetric space, we can find Killing vectors for which $\xi_\lambda(x)$ vanishes and for which $\xi_{\lambda;\kappa}(x)$ is an arbitrary antisymmetric matrix. It follows then that the co-

efficient of $\xi_{\lambda;\kappa}(x)$ in Eq. (13.1.12) must have a vanishing antisymmetric part, that is,

$$\begin{aligned} &-R^{\lambda}_{\rho\sigma\nu}\delta^{\kappa}_{\mu} + R^{\lambda}_{\mu\sigma\nu}\delta^{\kappa}_{\rho} - R^{\lambda}_{\sigma\rho\mu}\delta^{\kappa}_{\nu} + R^{\lambda}_{\nu\rho\mu}\delta^{\kappa}_{\sigma} \\ &= -R^{\kappa}_{\rho\sigma\nu}\delta^{\lambda}_{\mu} + R^{\kappa}_{\mu\sigma\nu}\delta^{\lambda}_{\rho} - R^{\kappa}_{\sigma\rho\mu}\delta^{\lambda}_{\nu} + R^{\kappa}_{\nu\rho\mu}\delta^{\lambda}_{\sigma} \end{aligned} \tag{13.2.1}$$

We also showed that at any given point x in a maximally symmetric space, there exist Killing vectors for which $\xi_{\lambda}(x)$ takes any values we like, so (13.1.12) and (13.2.1) require that

$$R^{\lambda}_{\sigma\rho\mu;\nu} = R^{\lambda}_{\nu\rho\mu;\sigma} \tag{13.2.2}$$

We actually only need to use (13.2.1), because we have shown in the last section that a space that is isotropic about every point, and hence satisfies (13.2.1), must also be homogeneous, and hence must also satisfy (13.2.2).

Our first step in the proof is to use Eq. (13.2.1) to derive a formula for the curvature tensor. Contracting κ with μ yields

$$-NR^{\lambda}_{\rho\sigma\nu} + R^{\lambda}_{\rho\sigma\nu} - R^{\lambda}_{\sigma\rho\nu} + R^{\lambda}_{\nu\rho\sigma} = -R^{\lambda}_{\rho\sigma\nu} + R_{\sigma\rho}\delta^{\lambda}_{\nu} - R_{\nu\rho}\delta^{\lambda}_{\sigma}$$

(Recall that $R^{\kappa}_{\kappa\sigma\nu}$ vanishes, $-R^{\kappa}_{\sigma\rho\kappa}$ is the Ricci tensor $R_{\sigma\rho}$, and in N dimensions, $\delta^{\kappa}_{\kappa} = N$.) Using the cyclic sum rule (6.6.5) and the antisymmetry of $R^{\lambda}_{\sigma\rho\nu}$, we find

$$(N-1)R_{\lambda\rho\sigma\nu} = R_{\nu\rho}g_{\lambda\sigma} - R_{\sigma\rho}g_{\lambda\nu} \tag{13.2.3}$$

But this must be antisymmetric in λ and ρ, so

$$R_{\nu\rho}g_{\lambda\sigma} - R_{\sigma\rho}g_{\lambda\nu} = -R_{\nu\lambda}g_{\rho\sigma} + R_{\sigma\lambda}g_{\rho\nu}$$

Contracting λ with ν, we find

$$R_{\sigma\rho} - NR_{\sigma\rho} = -R^{\lambda}{}_{\lambda}g_{\sigma\rho} + R_{\rho\sigma}$$

The Ricci tensor thus takes the form

$$R_{\sigma\rho} = \frac{1}{N}\, g_{\sigma\rho}R^{\lambda}{}_{\lambda} \tag{13.2.4}$$

Inserting this in (13.2.3) gives our formula for the curvature tensor

$$R_{\lambda\rho\sigma\nu} = \frac{R^{\lambda}{}_{\lambda}}{N(N-1)}\{g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu}\} \tag{13.2.5}$$

This formula satisfies (13.2.1), so there is nothing further to be learned from that condition.

In a space that is isotropic about *every* point, Eqs. (13.2.4) and (13.2.5) will hold everywhere, and we can use the Bianchi identities to say something about the dependence of the curvature scalar $R^{\lambda}{}_{\lambda}$ on position. Using (13.2.4) in (6.8.4), we have

$$0 = [R^{\sigma}{}_{\rho} - \tfrac{1}{2}\delta^{\sigma}{}_{\rho}R^{\lambda}{}_{\lambda}]_{;\sigma} = \left(\frac{1}{N} - \frac{1}{2}\right)(R^{\lambda}{}_{\lambda})_{;\sigma}$$

or

$$0 = \left(\frac{1}{N} - \frac{1}{2}\right) \frac{\partial}{\partial x^\sigma} R^\lambda{}_\lambda \tag{13.2.6}$$

Hence any space of three or more dimensions, in which (13.2.4) holds everywhere, will have $R^\lambda{}_\lambda$ *constant*. It is convenient to introduce a curvature constant K in place of $R^\lambda{}_\lambda$, with

$$R^\lambda{}_\lambda \equiv -N(N-1)K \tag{13.2.7}$$

Using this in (13.2.4) gives the Ricci tensor and the Riemann-Christoffel tensor here as

$$R_{\sigma\rho} = -(N-1)Kg_{\sigma\rho} \tag{13.2.8}$$

$$R_{\lambda\rho\sigma\nu} = K\{g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}\} \tag{13.2.9}$$

In differential geometry a space with these properties is called a *space of constant curvature*.

Incidentally, we showed in Section 6.7 that the curvature tensor in two dimensions is always of the form (13.2.5), so it is not surprising that in this case (13.2.6) does not allow us to draw any conclusions about the constancy of $R^\lambda{}_\lambda$. However, by using (13.2.2) one can show that the quantity K in (13.2.9) is also constant for maximally symmetric spaces of dimensionality $N = 2$.

Now suppose that we are given two metrics $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$, both having the same numbers of positive and negative eigenvalues, and both satisfying the condition (13.2.9) for a maximally symmetric space, that is,

$$R_{\lambda\rho\sigma\nu} = K(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}) \tag{13.2.10}$$

$$R'_{\lambda\rho\sigma\nu} = K(g'_{\sigma\rho}g'_{\lambda\nu} - g'_{\nu\rho}g'_{\lambda\sigma}) \tag{13.2.11}$$

with the same curvature constant K. We shall show that $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ must be equivalent, in the sense that there is a transformation $x \to x'$ that converts $g_{\mu\nu}(x)$ into $g'_{\mu\nu}(x')$, that is, for which

$$g'_{\mu\nu}(x') \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = g_{\rho\sigma}(x) \tag{13.2.12}$$

We shall prove this by actually constructing $x'^\mu(x)$ as a power series in x^μ. First, note that the equality in the numbers of positive and negative eigenvalues of $g_{\mu\nu}$ and $g'_{\mu\nu}$ means that we can find a nonsingular matrix $d^\mu{}_\rho$ for which

$$g'_{\mu\nu}(0)\, d^\mu{}_\rho\, d^\nu{}_\sigma = g_{\rho\sigma}(0) \tag{13.2.13}$$

(The argument here is the same as in Section 6.4.) Thus we can satisfy (13.2.12) to *zero* order in x with

$$x'^\mu = d^\mu{}_\rho x^\rho$$

Now we proceed by mathematical induction. Suppose that we succeed in satisfying (13.2.12) to order $n-1$ in x^μ with a polynomial

$$x'^\mu(x) = d^\mu{}_\rho x^\rho + \sum_{m=2}^{n} \frac{1}{m!} d^\mu{}_{\rho_1 \cdots \rho_m} x^{\rho_1} \cdots x^{\rho_m} \tag{13.2.14}$$

We want to add a term of order $n+1$ in x^μ so that (13.2.12) holds to order n. This condition will be satisfied if the derivative of (13.2.12) holds in order $n-1$. that is, if

$$\frac{\partial^2 x'^\mu}{\partial x^\rho \, \partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} g'_{\mu\nu}(x') + \frac{\partial^2 x'^\nu}{\partial x^\sigma \, \partial x^\lambda} \frac{\partial x'^\mu}{\partial x^\rho} g'_{\mu\nu}(x')$$

$$+ \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x'^\kappa}{\partial x^\lambda} \frac{\partial g'_{\mu\nu}(x')}{\partial x'^\kappa} = \frac{\partial g_{\rho\sigma}(x)}{\partial x^\lambda} \qquad \text{in order } x^{n-1}$$

This will be satisfied if (and, in fact, only if)

$$\frac{\partial^2 x'^\mu}{\partial x^\rho \, \partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} g'_{\mu\nu}(x') = g_{\sigma\tau}(x) \Gamma^\tau_{\lambda\rho}(x)$$

$$- \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x'^\kappa}{\partial x^\lambda} g'_{\nu\eta}(x') \Gamma'^\eta_{\mu\kappa}(x') \qquad \text{in order } x^{n-1}$$

This only needs to hold in order $n-1$ in x^μ, so we can use (13.2.12), which was assumed to hold to this order, to convert it into an equivalent requirement

$$\frac{\partial^2 x'^\mu}{\partial x^\rho \, \partial x^\lambda} = \frac{\partial x'^\mu}{\partial x^\kappa} \Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial x'^\kappa}{\partial x^\lambda} \Gamma'^\mu_{\nu\kappa}(x') \qquad \text{in order } x^{n-1} \tag{13.2.15}$$

We can use (13.2.14), which is correct to order x^n, to calculate the term on the right-hand side of order x^{n-1}. Let us write the result as

$$\left[\frac{\partial x'^\mu}{\partial x^\kappa} \Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial x'^\kappa}{\partial x^\lambda} \Gamma'^\mu_{\nu\kappa}(x') \right]_{\text{order } n-1}$$

$$= \frac{1}{(n-1)!} c^\mu_{\lambda\rho\sigma_1 \cdots \sigma_{n-1}} x^{\sigma_1} \cdots x^{\sigma_{n-1}} \tag{13.2.16}$$

the coefficients $c^\mu_{\lambda\rho} \ldots$ depending in a complicated way on the functions $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ and on the previously determined coefficients $d^\mu_{\rho_1 \cdots \rho_m}$. Then (13.2.15) will be satisfied in order $n-1$ if we add to (13.2.14) a term

$$[x'^\mu(x)]_{\text{order } n+1} = \frac{1}{(n+1)!} c^\mu_{\lambda\rho\sigma_1 \cdots \sigma_{n-1}} x^\lambda x^\rho x^{\sigma_1} \cdots x^{\sigma_{n-1}} \tag{13.2.17}$$

provided that the coefficient $c^{\mu}_{\lambda\rho\sigma_1\cdots\sigma_{n-1}}$ is totally symmetric in all its lower indices. These coefficients are obviously symmetric under interchange of λ and ρ or among the σ_m indices, so the only condition that needs to be satisfied is that they are symmetric between λ and any σ_m, or equivalently, that the derivative of (13.2.16) with respect to x^σ should be symmetric between λ and σ:

$$\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x'^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\,\Gamma'^\mu_{\nu\kappa}(x')\right)$$
$$= \frac{\partial}{\partial x^\lambda}\left(\frac{\partial x'^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\rho}(x) - \frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\sigma}\,\Gamma'^\mu_{\nu\kappa}(x')\right) \qquad \text{in order } x^{n-2} \tag{13.2.18}$$

Since (13.2.12) is assumed to hold to order x^{n-1}, its derivative, Eq. (13.2.15), will hold to order x^{n-2}, so we can use (13.2.12) and (13.2.15) to rewrite (13.2.18) as the equivalent requirement

$$\frac{\partial x'^\mu}{\partial x^\kappa}\,R^\kappa_{\rho\lambda\eta}(x) = \frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial x'^\sigma}{\partial x^\eta}\,R'^\mu_{\nu\kappa\sigma}(x') \qquad \text{in order } x^{n-2} \tag{13.2.19}$$

Now for the first time we use Eqs. (13.2.10) and (13.2.11), which allow (13.2.19) to be replaced with the equivalent requirement

$$\frac{\partial x'^\mu}{\partial x^\eta}\,g_{\lambda\rho}(x) - \frac{\partial x'^\mu}{\partial x^\lambda}\,g_{\rho\eta}(x)$$
$$= \frac{\partial x'^\nu}{\partial x^\rho}\left(\frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial x'^\mu}{\partial x^\eta}\,g'_{\nu\kappa}(x') - \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\sigma}{\partial x^\eta}\,g'_{\nu\sigma}(x')\right) \qquad \text{in order } x^{n-2} \tag{13.2.20}$$

This condition *is* satisfied, because (13.2.12) was assumed to hold to order x^{n-1}. To recapitulate, this implies that (13.2.19) holds in order x^{n-2}, which implies that (13.2.18) holds in order x^{n-2}, which implies that the coefficients $c^{\mu}_{\lambda\rho\sigma_1\cdots\sigma_m}$ are totally symmetric in their lower indices, which implies that (13.2.17) satisfies (13.2.15), which implies that by adding (13.2.17) to (13.2.14) we can satisfy (13.2.12) to order x^n. Thus, if (13.2.12) can be satisfied to order x^{n-1} by a polynomial $x'(x)$ of order n, it can be satisfied to order x^n by a polynomial $x'(x)$ of order $n+1$, and therefore a function $x'(x)$ satisfying (13.2.12) exactly can be built up as a power series, as was to be proven.

3 Maximally Symmetric Spaces: Construction

Maximally symmetric spaces are essentially unique, so we can learn all about them by constructing examples with arbitrary curvature K in any way we like.

This is one rather obvious way to carry out this construction. (See Figure 13.1.) Consider a flat $(N + 1)$-dimensional space, with metric given by

$$-d\tau^2 \equiv g_{AB}\, dx^A\, dx^B = C_{\mu\nu}\, dx^\mu\, dx^\nu + K^{-1}\, dz^2 \tag{13.3.1}$$

where $C_{\mu\nu}$ is a constant $N \times N$ matrix and K is some constant. We can embed a non-Euclidean N-dimensional space in this larger space by restricting the variables x^μ and z to the surface of a sphere (or pseudosphere):

$$KC_{\mu\nu}x^\mu x^\nu + z^2 = 1 \tag{13.3.2}$$

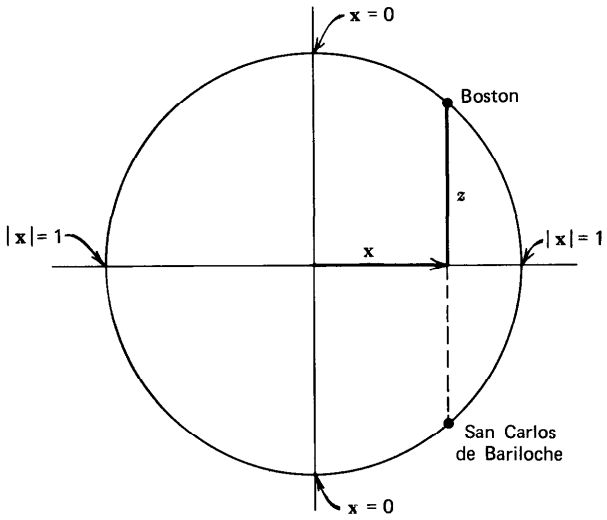

Figure 13.1 Representation of points on a sphere by projection onto the equatorial plane. Note that two points on the sphere correspond to each projected point with given coordinates x^i.

On this surface, dz^2 is given by

$$dz^2 = \frac{K^2(C_{\mu\nu}x^\mu\, dx^\nu)^2}{z^2}$$

$$= \frac{K^2(C_{\mu\nu}x^\mu\, dx^\nu)^2}{(1 - KC_{\mu\nu}x^\mu x^\nu)}$$

and therefore (13.3.1) gives

$$-d\tau^2 = C_{\mu\nu}\, dx^\mu\, dx^\nu + \frac{K(C_{\mu\nu}x^\mu\, dx^\nu)^2}{(1 - KC_{\rho\sigma}x^\rho x^\sigma)} \tag{13.3.3}$$

The metric is then

$$g_{\mu\nu}(x) = C_{\mu\nu} + \frac{K}{(1 - KC_{\rho\sigma}x^\rho x^\sigma)} C_{\mu\lambda}x^\lambda C_{\nu\kappa}x^\kappa \tag{13.3.4}$$

A flat space appears here as the special case $K = 0$.

This construction makes it obvious that (13.3.4) admits an $(N(N + 1)/2)$-parameter group of isometries, for both the $(N + 1)$-dimensional line element (13.3.1) and the "embedding" condition (13.3.2) are manifestly invariant under rigid "rotations" of the $(N + 1)$-dimensional space, that is, under the transformations

$$x^\mu \to x'^\mu = R^\mu{}_\nu x^\nu + R^\mu{}_z z \tag{13.3.5}$$

$$z \to z' = R^z{}_\mu z^\mu + R^z{}_z z \tag{13.3.6}$$

where the $R^A{}_B$ are constants, with

$$C_{\mu\nu}R^\mu{}_\rho R^\nu{}_\sigma + K^{-1}R^z{}_\rho R^z{}_\sigma = C_{\rho\sigma} \tag{13.3.7}$$

$$C_{\mu\nu}R^\mu{}_\rho R^\nu{}_z + K^{-1}R^z{}_\rho R^z{}_z = 0 \tag{13.3.8}$$

$$C_{\mu\nu}R^\mu{}_z R^\nu{}_z + K^{-1}(R^z{}_z)^2 = K^{-1} \tag{13.3.9}$$

It is convenient to distinguish two classes of simple transformations satisfying (13.3.7)–(13.3.9):

$$\text{(A)} \qquad R^\mu{}_\nu = \mathscr{R}^\mu{}_\nu \qquad R^\mu{}_z = R^z{}_\mu = 0 \qquad R^z{}_z = 1 \tag{13.3.10}$$

where $\mathscr{R}^\mu{}_\nu$ is any $N \times N$ matrix with

$$C_{\mu\nu}\mathscr{R}^\mu{}_\rho \mathscr{R}^\nu{}_\sigma = C_{\rho\sigma} \tag{13.3.11}$$

These are just rigid "rotations" about the origin:

$$x'^\mu = \mathscr{R}^\mu{}_\nu x^\nu \tag{13.3.12}$$

$$\text{(B)} \qquad R^\mu{}_z = a^\mu \qquad R^z{}_\mu = -KC_{\mu\nu}a^\nu \qquad R^z{}_z = (1 - KC_{\rho\sigma}a^\rho a^\sigma)^{1/2} \tag{13.3.13}$$

$$R^\mu{}_\nu = \delta^\mu{}_\nu - bKC_{\nu\rho}a^\rho a^\mu \tag{13.3.14}$$

where a^μ is arbitrary except that $R^z{}_z$ must be real, that is,

$$KC_{\rho\sigma}a^\rho a^\sigma \leq 1 \tag{13.3.15}$$

and

$$b \equiv \frac{1 - (1 - KC_{\rho\sigma}a^\rho a^\sigma)^{1/2}}{KC_{\rho\sigma}a^\rho a^\sigma} \tag{13.3.16}$$

These are "quasitranslations," with

$$x'^\mu = x^\mu + a^\mu[(1 - KC_{\rho\sigma}x^\rho x^\sigma)^{1/2} - bKC_{\rho\sigma}x^\rho a^\sigma] \tag{13.3.17}$$

In particular, these transformations take the origin $x^\mu = 0$ into a^μ.

The existence of isometries (13.3.17) that take the origin into any point (at least within a finite region) means that this space is *homogeneous*; any point is geometrically like any other point. (Our coordinate system hides this property, just as a polar projection map of the earth hides the fact that the curvature of the earth is about the same in Massachusetts as at the North Pole.) Also, the existence of isometries (13.3.10) that include all rigid "rotations" about the origin means that this space is *isotropic* about the origin. Since the metric is homogeneous, and isotropic about the origin, it is isotropic about every point, and maximally symmetric.

We can construct the Killing vectors for this metric by letting the finite transformations (13.2.5), (13.2.6) approach the unit transformation. First, consider the transformations (A), and let

$$\mathscr{R}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon\Omega^{\mu}{}_{\nu}, \qquad |\varepsilon| \ll 1$$

$$C_{\mu\sigma}\Omega^{\mu}{}_{\rho} + C_{\rho\mu}\Omega^{\mu}{}_{\sigma} = 0 \tag{13.3.18}$$

Comparing with (13.1.3), the corresponding Killing vectors are

$$\xi^{\mu}{}_{\Omega}(x) = \Omega^{\mu}{}_{\nu}x^{\nu} \tag{13.3.19}$$

Next, consider the transformations (B), and let

$$a^{\mu} = \varepsilon\alpha^{\mu}, \qquad |\varepsilon| \ll 1$$

Comparing with (13.1.3), the corresponding Killing vectors are

$$\xi^{\mu}{}_{\alpha}(x) = \alpha^{\mu}[1 - KC_{\mu\nu}x^{\mu}x^{\nu}]^{1/2} \tag{13.3.20}$$

The reader may check that (13.3.19) and (13.3.20) do satisfy the Killing conditions (13.1.5). There are $N(N-1)/2$ independent parameters $\Omega^{\mu}{}_{\nu}$ [that is, N^2 elements $\Omega^{\mu}{}_{\nu}$, subject to the $N(N+1)/2$ conditions (13.3.18)] and N parameters α^{μ}, so this metric admits $N(N+1)/2$ independent Killing vectors, verifying maximal symmetry.

The geodesics of this metric take a remarkably simple form. From (13.3.4) we can readily calculate that the affine connection is

$$\Gamma^{\mu}_{\nu\lambda} = Kx^{\mu}g_{\nu\lambda} \tag{13.3.21}$$

so the differential equation for a geodesic is

$$\frac{d^2x^{\mu}}{d\tau^2} + Kx^{\mu} = 0 \tag{13.3.22}$$

The solutions are thus linear combinations of $\sin(\tau\sqrt{K})$ and $\cos(\tau\sqrt{K})$ for $K > 0$, or of $\sinh(\tau\sqrt{-K})$ and $\cosh(\tau\sqrt{-K})$ for $K < 0$.

We can uncover the inner properties of this space by calculating the curvature

tensor; a straightforward computation using Eqs. (6.6.2) and (13.2.21) gives the Riemann-Christoffel tensor for the metric (13.3.4) as

$$R_{\kappa\nu\rho\sigma} = K[C_{\kappa\sigma}C_{\nu\rho} - C_{\kappa\rho}C_{\nu\sigma}] + K^2[1 - KC_{\mu\lambda}x^\mu x^\lambda]^{-1}[C_{\kappa\sigma}x_\nu x_\rho - C_{\kappa\rho}x_\nu x_\sigma + C_{\nu\rho}x_\kappa x_\sigma - C_{\nu\sigma}x_\rho x_\kappa]$$

(where $X_\nu \equiv C_{\nu\mu}X^\mu$), or

$$R_{\kappa\nu\rho\sigma} = K[g_{\rho\nu}g_{\kappa\sigma} - g_{\sigma\nu}g_{\kappa\rho}]$$

in agreement with Eq. (13.2.9). Hence the constant K introduced in Eqs. (13.3.1) and (13.3.2) is the same as the curvature constant introduced in the last section.

Since K is an invariant parameter, we cannot by a coordinate transformation convert the metric (13.3.4) into a similar metric with a different K. In contrast, Eq. (13.3.3) makes it obvious that by a linear transformation

$$x^\mu = A^\mu{}_\nu x'^\nu$$

we can convert the metric (13.3.4) into a similar metric with the same K and with $C_{\mu\nu}$ changed into

$$C'_{\mu\nu} = A^\rho{}_\mu A^\sigma{}_\nu C_{\rho\sigma}$$

Our discussion in Section 3.6 shows that in this way $C_{\mu\nu}$ can be changed into any real symmetric matrix we like, as long as we do not change the numbers of its positive and negative eigenvalues. Also, the numbers of eigenvalues of each sign of the matrix $C_{\mu\nu}$ are the same as for the matrix $g_{\mu\nu}$ at the point $x = 0$, and hence the same everywhere, since all points are equivalent.

An N-dimensional metric that allows the introduction of locally Euclidean (as opposed, say, to Minkowskian) coordinate systems will have all its eigenvalues positive, so for $K \neq 0$ we can take $C_{\mu\nu}$ as $|K|^{-1}$ times the unit matrix, in which case (13.3.3) becomes

$$ds^2 = K^{-1}\left[d\mathbf{x}^2 + \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1 - \mathbf{x}^2}\right] \quad \text{for } K > 0 \tag{13.3.23}$$

or

$$ds^2 = |K|^{-1}\left[d\mathbf{x}^2 - \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1 + \mathbf{x}^2}\right] \quad \text{for } K < 0 \tag{13.3.24}$$

For $K = 0$, we take $C_{\mu\nu}$ as just the unit matrix, and (13.3.3) gives

$$ds^2 = d\mathbf{x}^2 \qquad \text{for } K = 0 \tag{13.3.25}$$

(We are using an obvious N-dimensional vector notation. Also, we have replaced $-d\tau^2$ with a proper length ds^2, because for the moment we are doing geometry rather than physics.) Let us explore the *global* properties of these spaces.

For $K > 0$, our most convenient approach is to go back to the interpretation

of (13.3.23) as the metric of the curved space embedded by Eq. (13.3.2) in the flat space (13.3.1); that is, (13.3.23) describes the surface

$$\mathbf{x}^2 + z^2 = 1 \tag{13.3.26}$$

in the flat space with

$$ds^2 = K^{-1}[d\mathbf{x}^2 + dz^2] \tag{13.3.27}$$

Obviously this metric simply describes the surface of a sphere of radius $K^{-1/2}$ in an $(N + 1)$-dimensional Euclidean space. (To make the coordinates $\mathbf{x}$ and z truly Euclidean, we should define $\mathbf{x}' = K^{-1/2}\mathbf{x}$ and $z' = K^{-1/2}z$, in which case (13.3.26) reads $\mathbf{x}'^2 + z'^2 = K^{-1}$.) Indeed, in two dimensions we can introduce angular coordinates θ, φ by:

$$x_1 = \sin\theta \cos\varphi \qquad x_2 = \sin\theta \sin\varphi$$

and (13.3.27) then becomes the familiar line element on a sphere of radius $K^{-1/2}$:

$$ds^2 = K^{-1}[d\theta^2 + \sin^2\theta\, d\varphi^2] \tag{13.3.28}$$

In general, the range of the variables $\mathbf{x}$ is

$$\mathbf{x}^2 \leq 1$$

However, each $\mathbf{x}$ actually corresponds to *two* points, corresponding to the two roots of Eq. (13.3.26) for z. (For instance, in two dimensions the components of $\mathbf{x}$ are the coordinates of points on a sphere projected on a tangent plane; in a polar projection map of the earth, Boston will appear at the same point as San Carlos de Bariloche, Argentina.) The volume of the N-dimensional space described by (13.3.23) is therefore

$$V_N = 2\int_{\mathbf{x}^2 \leq 1} \sqrt{g}\, dx_1 \cdots dx_N = 2K^{-N/2} \int_{\mathbf{x}^2 \leq 1} \frac{dx_1 \cdots dx_N}{[1 - \mathbf{x}^2]^{1/2}}$$

A straightforward calculation gives

$$V_N = \frac{2\pi^{(N+1)/2}}{\Gamma((N+1)/2)} K^{-N/2} \tag{13.3.29}$$

For instance, $V_1 = 2\pi K^{-1/2}$, which is just the perimeter of a circle of radius $K^{-1/2}$, and $V_2 = 4\pi K^{-1}$, which is just the area of a sphere with radius $K^{-1/2}$. A three-dimensional space of constant positive curvature has the volume

$$V_3 = 2\pi^2 K^{-3/2}$$

We can also calculate the circumference of such spaces, using for the geodesics the solutions of Eq. (13.3.22), which now reads

$$\frac{d^2\mathbf{x}}{ds^2} + K\mathbf{x} = 0 \tag{13.3.30}$$

The solutions that pass through the point $\mathbf{x} = 0$ are

$$\mathbf{x} = \mathbf{e} \sin(sK^{1/2}) \tag{13.3.31}$$

where, in order to satisfy (13.3.23),

$$\mathbf{e}^2 = 1 \tag{13.3.32}$$

As we go out along a geodesic from the "North pole" $\mathbf{x} = 0$, we reach the "equator" $\mathbf{x} = \mathbf{e}$ at $s = \pi K^{-1/2}/2$, we reach the "South pole" $\mathbf{x} = 0$ at $s = \pi K^{-1/2}$, we reach the opposite point $\mathbf{x} = -\mathbf{e}$ of the "equator" at $s = 3\pi K^{-1/2}/2$, and we return to our starting point at $s = 2\pi K^{-1/2}$. Thus the distance from any point around the whole space and back to itself along a geodesic is

$$L = 2\pi K^{-1/2} \tag{13.3.33}$$

for spaces of constant positive curvature and arbitrary dimensionality. This calculation shows very clearly that the space described by (13.3.23) is *finite*, but it is not *bounded*; when we come to the apparent singularity at $\mathbf{x}^2 = 1$, we continue right through, but with z given by the root of Eq. (13.3.26) of opposite sign.

For $K < 0$ the metric (13.3.24) does not even have an apparent singularity, and there is nothing to restrict the coordinates $\mathbf{x}$ to any finite range. This can be seen even more definitely by calculating the geodesics, which are now given by Eqs. (13.3.30) and (13.3.24) as

$$\mathbf{x} = \mathbf{e} \sinh(s(-K)^{1/2}) \tag{13.3.34}$$

$$\mathbf{e}^2 = 1 \tag{13.3.35}$$

We can obviously go out along this geodesic an unlimited distance from the origin. For $N = 2$, this space is just that discovered by Gauss, Bólyai, and Lobachevski. [See Section 1.1. In order to put the metric in the form (1.1.9) of Klein's model, it is necessary to introduce a new set of coordinates x'^i, defined by $\mathbf{x}' = \mathbf{x}(1 + \mathbf{x}^2)^{-1/2}$.] We see from (13.3.1) and (13.3.2) that this geometry describes the surface

$$-\mathbf{x}^2 + z^2 = 1 \tag{13.3.36}$$

in a flat space, with

$$ds^2 = |K|^{-1}[d\mathbf{x}^2 - dz^2] \tag{13.3.37}$$

The minus sign in (13.3.37) means that this flat space is *not* Euclidean. It is therefore understandable that the Gauss-Bólyai-Lobachevski geometry could not be discovered until geometers had learned to think of curved surfaces, not as subspaces of an ordinary Euclidean space, but as spaces characterized by their own inner metric relations.

Finally, let us return to space-time, and consider the structure of a four-dimensional maximally symmetric metric with three positive and one negative eigenvalue. In this case, we can set

$$C_{\mu\nu} = \eta_{\mu\nu} \tag{13.3.38}$$

and the metric is

$$-d\tau^2 = d\mathbf{x}^2 - dt^2 + \frac{K(\mathbf{x}\cdot d\mathbf{x} - t\,dt)^2}{1 - K(\mathbf{x}^2 - t^2)} \tag{13.3.39}$$

For $K > 0$, we can introduce coordinates in which the metric appears *spatially* flat, by setting

$$t = \frac{1}{\sqrt{K}}\left[\frac{Kr'^2}{2}\cosh(K^{1/2}t') + \left(1 + \frac{Kr'^2}{2}\right)\sinh(K^{1/2}t')\right]$$
$$\mathbf{x} = \mathbf{x}'\exp(K^{1/2}t') \tag{13.3.40}$$

Then (13.3.39) becomes

$$d\tau^2 = dt'^2 - \exp(2K^{1/2}t')\,d\mathbf{x}'^2 \tag{13.3.41}$$

We can also introduce coordinates in which the metric appears time-independent, by setting

$$t'' = t' - \frac{1}{2K^{1/2}}\ln\left[1 - K\mathbf{x}'^2\exp(2K^{1/2}t')\right]$$
$$\mathbf{x}'' = \mathbf{x}'\exp(K^{1/2}t') \tag{13.3.42}$$

Then (13.3.41) becomes

$$d\tau^2 = (1 - K\mathbf{x}''^2)\,dt''^2 - d\mathbf{x}''^2 - \frac{K(\mathbf{x}''\cdot d\mathbf{x}'')^2}{1 - K\mathbf{x}''^2} \tag{13.3.43}$$

This metric was first discussed in this form by deSitter;[2] it will provide the basis for our treatment of the steady state cosmology in Chapter 14.

Once again, it should be stressed that the maximally symmetric metric (13.3.4), although derived by an apparently arbitrary procedure, actually represents the most general possible maximally symmetric metric, because the uniqueness theorem of the last section tells us that any other maximally symmetric metric can be converted into the form (13.3.4) by a suitable coordinate transformation.

4 Tensors in a Maximally Symmetric Space

The assumption of maximal symmetry can be applied, not only to the metric of a space, but to any tensor fields that inhabit the space. A tensor field $T_{\mu\nu}\ldots$

is said to be *form-invariant* under a transformation $x \to x'$ if $T'_{\mu\nu}\ldots(x')$ is the same function of its argument x'^{μ} as $T_{\mu\nu}\ldots(x)$ was of its argument x^{μ}, that is,

$$T'_{\mu\nu}\ldots(y) = T_{\mu\nu}\ldots(y) \qquad \text{for all } y \tag{13.4.1}$$

At any given point, the transformed tensor is given by the usual formula

$$T_{\mu\nu}\ldots(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \cdots T'_{\rho\sigma}\ldots(x')$$

so the form-invariance condition (13.4.1) reads

$$T_{\mu\nu}\ldots(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \cdots T_{\rho\sigma}\ldots(x') \tag{13.4.2}$$

For an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}(x) \qquad |\varepsilon| \ll 1$$

the condition (13.4.2) becomes, to first order in ε,

$$0 = \frac{\partial\xi^{\rho}(x)}{\partial x^{\mu}} T_{\rho\nu}\ldots(x) + \frac{\partial\xi^{\sigma}(x)}{\partial x^{\nu}} T_{\mu\sigma}\ldots(x) + \cdots + \xi^{\lambda}(x) \frac{\partial}{\partial x^{\lambda}} T_{\mu\nu}\ldots(x) \tag{13.4.3}$$

(That is, the *Lie derivative* of $T_{\mu\nu}\ldots$ with respect to ξ^{λ} vanishes; see Section 10.9.) A tensor in a maximally symmetric space, which satisfies (13.4.3) for all $N(N+1)/2$ independent Killing vectors $\xi^{\lambda}(x)$, will be called *maximally form-invariant.*

For a scalar $S(x)$, Eq. (13.4.3) reads simply

$$\xi^{\lambda}(x) \frac{\partial}{\partial x^{\lambda}} S(x) = 0 \tag{13.4.4}$$

If the scalar is maximally form-invariant, then $\xi^{\lambda}(x)$ can at any given point be chosen to have any value we like, and (13.4.4) therefore requires that S be constant:

$$\frac{\partial S}{\partial x^{\lambda}} = 0 \tag{13.4.5}$$

For any other maximally form-invariant tensor, it is convenient first to choose a Killing vector $\xi^{\lambda}(x)$ that at a given point X satisfies

$$\xi^{\lambda}(X) = 0$$

and for which the quantities

$$\xi_{\sigma;\mu}(X) = g_{\sigma\rho}(X) \left(\frac{\partial\xi^{\rho}(x)}{\partial x^{\mu}} \right)_{x=X}$$

form an arbitrary antisymmetric matrix. Equation (13.4.3) then reads, at $x = X$:

$$0 = \xi_{\sigma;\tau}\{\delta^{\tau}_{\mu}T^{\sigma}{}_{\nu}\ldots + \delta^{\tau}_{\nu}T_{\mu}{}^{\sigma}\ldots + \cdots\}$$

Since $\xi_{\sigma;\tau}$ is an arbitrary antisymmetric matrix, its coefficient must be symmetric in σ and τ:

$$\delta^{\tau}_{\mu}T^{\sigma}{}_{\nu}\ldots + \delta^{\tau}_{\nu}T_{\mu}{}^{\sigma}\ldots + \cdots = \delta^{\sigma}_{\mu}T^{\tau}{}_{\nu}\ldots + \delta^{\sigma}_{\nu}T_{\mu}{}^{\tau}\ldots + \cdots \tag{13.4.6}$$

Since X was arbitrary, this must hold everywhere.

For a maximally form-invariant vector $A_{\mu}(x)$, Eq. (13.4.6) reads

$$\delta^{\tau}_{\mu}A^{\sigma} = \delta^{\sigma}_{\mu}A^{\tau}$$

Contracting τ with μ, we find that in N dimensions

$$NA^{\sigma} = A^{\sigma}$$

so, except for the trivial case $N = 1$, we must have

$$A^{\sigma} = 0 \tag{13.4.7}$$

For a maximally form-invariant tensor $B_{\mu\nu}$ of second rank, Eq. (13.4.6) reads

$$\delta_{\mu}{}^{\tau}B^{\sigma}{}_{\nu} + \delta_{\nu}{}^{\tau}B_{\mu}{}^{\sigma} = \delta_{\mu}{}^{\sigma}B^{\tau}{}_{\nu} + \delta^{\sigma}{}_{\nu}B_{\mu}{}^{\tau}$$

Contracting τ with μ gives

$$NB^{\sigma}{}_{\nu} + B_{\nu}{}^{\sigma} = B^{\sigma}{}_{\nu} + \delta^{\sigma}{}_{\nu}B_{\mu}{}^{\mu}$$

or, lowering the σ index,

$$(N - 1)B_{\sigma\nu} + B_{\nu\sigma} = g_{\sigma\nu}B_{\mu}{}^{\mu} \tag{13.4.8}$$

Subtracting the same equation with ν and σ interchanged yields

$$(N - 2)(B_{\sigma\nu} - B_{\nu\sigma}) = 0$$

so as long as $N \neq 2$, the tensor $B_{\sigma\nu}$ must be symmetric:

$$B_{\sigma\nu} = B_{\nu\sigma} \tag{13.4.9}$$

(In two dimensions, $B_{\sigma\nu}$ can have an antisymmetric part proportional to $g^{-1/2}\varepsilon_{\sigma\nu}$; see Section 4.4.) Using (13.4.9) in (13.4.8) gives now for $N \geq 3$ (and for the symmetric part of $B_{\sigma\nu}$ for $N = 2$)

$$B_{\sigma\nu} = fg_{\sigma\nu} \tag{13.4.10}$$

where

$$f \equiv \frac{1}{N}B_{\mu}{}^{\mu}$$

To determine the dependence of f on the coordinates, we can use (13.4.10) back in the form-invariance condition (13.4.3):

$$0 = \frac{\partial \xi^\rho}{\partial x^\mu} f g_{\rho\nu} + \frac{\partial \xi^\sigma}{\partial x^\nu} f g_{\mu\sigma} + \xi^\lambda \frac{\partial}{\partial x^\lambda}(f g_{\mu\nu})$$

But $g_{\mu\varphi}$ satisfies the Killing condition (13.1.4), so this becomes

$$0 = g_{\mu\nu}\xi^\lambda \frac{\partial f}{\partial x^\lambda}$$

In a maximally symmetric space we can at any given point choose ξ^λ to have any value we like, and therefore

$$\frac{\partial f}{\partial x^\lambda} = 0 \tag{13.4.11}$$

Thus *the only maximally form-invariant tensor of second rank is the metric tensor, times a possible constant.*

5 Spaces with Maximally Symmetric Subspaces

In many cases of physical importance, the whole space (or space-time) is not maximally symmetric, but it can be decomposed into maximally symmetric subspaces. For instance, a spherically symmetric three-dimensional space can be decomposed into a family of spherical surfaces centered on the origin, each of which is described by a metric of the form (13.3.28). Also, in Chapter 14 we shall deal with space-times in which the metric is spherically symmetric and homogeneous on each "plane" of constant time.

We shall see here that the maximal symmetry of a family of subspaces imposes very strong constraints on the metric of the whole space. In order to state and prove these results, let us first adopt a suitable coordinate system. If the whole space has N dimensions and its maximally symmetric subspaces have M dimensions, then we can distinguish these subspaces from each other with $N - M$ coordinate labels v^a, and locate points within each subspace with M coordinates u^i. Some illustrations are given in Table 13.1.

Table 13.1 Examples of Spaces with Maximally Symmetric Subspaces

Example	v-Coordinates	u-Coordinates
Spherically symmetric space	r	θ, φ
Spherically symmetric space-time	r, t	θ, φ
Spherically symmetric and homogeneous space-time	t	r, θ, φ

We say that the subspaces with constant v^a are maximally symmetric if the metric of the whole space is invariant under a group of infinitesimal transformations

$$u^i \to u'^i = u^i + \varepsilon\xi^i(u, v) \tag{13.5.1}$$

$$v^a \to v'^a = v^a \tag{13.5.2}$$

with $M(M + 1)/2$ independent Killing vectors ξ^i. These are transformations of the general form (13.1.3), but with the special feature that the v^a are invariant, so that

$$\xi^a(u, v) = 0 \tag{13.5.3}$$

Note that, although these transformations affect only the u-variables, there is no reason why the transformation rules cannot depend parametrically on the labels v^a of the particular subspace being transformed. Also, our statement that there are $M(M + 1)/2$ "independent" Killing vectors should be construed to mean that there are $M(M + 1)/2$ Killing vectors that are not subject to any linear relations with u-independent coefficients.

The general result that governs the structure of such spaces is contained in the following theorem: It is always possible to choose the u-coordinates so that the metric of the whole space is given by

$$-d\tau^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu = g_{ab}(v)\,dv^a\,dv^b + f(v)\tilde{g}_{ij}(u)\,du^i\,du^j \tag{13.5.4}$$

where $g_{ab}(v)$ and $f(v)$ are functions of the v-coordinates alone, and $\tilde{g}_{ij}(u)$ is a function of the u-coordinates alone that is by itself the metric of an M-dimensional maximally symmetric space. (The summation convention is in force, with $a, b, \ldots$ running over the $N - M$ labels of the v-coordinates, and $i, j, k, l, \ldots$ running over the M labels of the u-coordinates.)

To begin our proof, we set down the condition that (13.5.1) is an isometry of the whole metric $g_{\mu\nu}(x)$. It is convenient to use this condition here in its original form (13.1.4) rather than in the more elegant covariant form (13.1.5). Each index $\mu, \nu, \rho \ldots$ in (13.1.4) now runs over the $N - M$ coordinate labels of the v^a and the M coordinate labels of the u^i, so (13.1.4) now yields three separate equations: For $\rho = i, \sigma = j$ we have

$$0 = \frac{\partial\xi^k(u, v)}{\partial u^i}\,g_{kj}(u, v) + \frac{\partial\xi^k(u, v)}{\partial u^j}\,g_{ki}(u, v) + \xi^k(u, v)\,\frac{\partial g_{ij}(u, v)}{\partial u^k} \tag{13.5.5}$$

for $\rho = i, \sigma = a$ we have

$$0 = \frac{\partial\xi^k(u, v)}{\partial u^i}\,g_{ka}(u, v) + \frac{\partial\xi^k(u, v)}{\partial v^a}\,g_{ik}(u, v) + \xi^k(u, v)\,\frac{\partial g_{ia}(u, v)}{\partial u^k} \tag{13.5.6}$$

and for $\rho = a$, $\sigma = b$ we have

$$0 = \frac{\partial \xi^k(u, v)}{\partial v^a} g_{kb}(u, v) + \frac{\partial \xi^k(u, v)}{\partial v^b} g_{ka}(u, v) + \xi^k(u, v) \frac{\partial g_{ab}(u, v)}{\partial u^k} \tag{13.5.7}$$

Of these three equations, the first simply tells us that $g_{ij}(u, v)$ must, for each fixed set of v^a, be the metric of an M-dimensional space, with coordinates u^i, which admits the Killing vector ξ^i. We assume here that there are $M(M + 1)/2$ such independent Killing vectors, so this means that the submatrix $g_{ij}(u, v)$ is by itself a maximally symmetric metric for each set of fixed v^a. According to the arguments of Section 13.1, it also follows that at any given point u_0 we can find Killing vectors $\xi^k(u, v)$ for which $\xi^k(u_0, v)$ and $\xi_{k;l}(u_0, v)$ take arbitrary values, subject only to the requirement that $\xi_{k;l} = -\xi_{l;k}$. Thus the metric $g_{ij}(u, v)$ is for each v both homogeneous in u and isotropic about any point.

The other two equations contain information about the other elements g_{ai} and g_{ab}, and also about the v-dependence of the Killing vectors. (This v-dependence is not entirely arbitrary. For instance, it is true that by redefining the u-coordinates we can always arrange that the metric $g_{ij}(u, v)$ has v-independent Killing vectors $\bar{\xi}^i(u)$, but the Killing vectors $\xi^i(u, v)$ of the whole space will then in general be linear combinations of the $\bar{\xi}^i(u)$, with coefficients that can depend on the v-coordinates.) In order to disentangle the different information contained in (13.5.6) and (13.5.7), it is extremely useful to choose a new set of coordinates $u'^i(u, v)$ of the maximally symmetric subspaces, so that g'_{ja} vanishes. Suppose for a moment that we can find a function $U^k(v; u_0)$ that satisfies the differential equation

$$g_{ik}(U, v) \frac{\partial U^k}{\partial v^a} = -g_{ia}(U, v) \tag{13.5.8}$$

with the initial condition that

$$U^k(v_0; u_0) \equiv {u_0}^k \tag{13.5.9}$$

at some point ${v_0}^a$. The coordinates u'^i, v'^a are then defined by

$$u^i = U^i(v'; u') \tag{13.5.10}$$

$$v^a = v'^a \tag{13.5.11}$$

In this coordinate system, the metric has

$$\begin{aligned} g'_{ja}(u', v') &= \frac{\partial u^l}{\partial u'^j} \frac{\partial u^k}{\partial v'^a} g_{lk}(u, v) + \frac{\partial u^l}{\partial u'^j} g_{la}(u, v) \\ &= \frac{\partial U^l(v'; u')}{\partial u'^j} \left\{ \frac{\partial U^k(v'; u')}{\partial v'^a} g_{lk}(U, v') + g_{la}(U, v') \right\} \end{aligned}$$

and thus (13.5.8) gives

$$g'_{ia} = 0 \tag{13.5.12}$$

Thus we can construct u'-coordinates in which g'_{ia} vanishes, if we can find solutions of the differential equation (13.5.8) with arbitrary initial conditions (13.5.9).

We can rewrite (13.5.8) in the equivalent form

$$\frac{\partial U^k}{\partial v^a} = -F^k{}_a(U, v) \tag{13.5.13}$$

where

$$F^k{}_a(U, v) \equiv \bar{g}^{ki}(U, v) g_{ia}(U, v) \tag{13.5.14}$$

and $\bar{g}^{ij}$ is the matrix reciprocal to g_{ij}, that is,

$$\bar{g}^{ij} g_{jk} = \delta^i_k \tag{13.5.15}$$

(The bar is to remind us that the ij-element $\bar{g}^{ij}$ of the matrix reciprocal to g_{ij} is not the same as the ij-element g^{ij} of the matrix $g^{\mu\nu}$ reciprocal to $g_{\mu\nu}$.) When there is only one v-coordinate, as will be the case in our chapters on cosmology, it is obvious that (13.5.13) can be solved with arbitrary initial conditions. In the general case, we shall have to do some work to prove that (13.5.13) is integrable. Our method is the same as in Section 13.2; we try to solve (13.5.13) in a neighborhood of v_0 with a power series in $v - v_0$:

$$U^k = \sum_{n=0}^{\infty} \frac{1}{n!} c^k{}_{a_1 \cdots a_n} (v - v_0)^{a_1} \cdots (v - v_0)^{a_n} \tag{13.5.16}$$

Clearly the initial conditions (13.5.9) are satisfied if we choose the $n = 0$ coefficient as

$$c^k = u_0{}^k$$

and Eq. (13.5.13) is satisfied to zero order in $v - v_0$ if we choose

$$c^k{}_a = -F^k{}_a(u_0, v_0)$$

Now, proceeding by mathematical induction, suppose that we are able to choose the terms in (13.5.16) up to order $(v - v_0)^n$ so that (13.5.13) is satisfied to order $(v - v_0)^{n-1}$. Then we can use these terms to calculate the term in $F^k{}_a(U, v)$ of order $(v - v_0)^n$. Let us write this term as

$$[F^k{}_a(U(v; u_0), v)]_{\text{order } n} = \frac{1}{n!} f^k_{ab_1 \cdots b_n} (v - v_0)^{b_1} \cdots (v - v_0)^{b_n}$$

Then (13.5.16) will satisfy (13.5.13) to order $(v - v_0)^n$ if we choose the term in U of order $n + 1$ as

$$[U^k(v; u_0)]_{n+1} = \frac{1}{(n+1)!} f^k_{ab_1 \cdots b_n} (v - v_0)^a (v - v_0)^{b_1} \cdots (v - v_0)^{b_n}$$

providing that f is symmetric in all its subscripts. Since $f^k_{ab_1 \cdots b_n}$ can obviously be chosen symmetric in the b's, it is sufficient to require that it should also be symmetric between a and any b, or equivalently that

$$\left[\frac{\partial}{\partial v^b} F^k{}_a(U(v; u_0), v)\right]_{\text{order } n-1}$$

should be symmetric in a and b. But U is assumed to satisfy (13.5.13) to order $(v - v_0)^{n-1}$, so this condition is satisfied if

$$\left[-\frac{\partial F^k{}_a(u, v)}{\partial u^l} F^l{}_b(u, v) + \frac{\partial F^k{}_a(u, v)}{\partial v^b} \right]_{u=U(v;u_0)}$$

is symmetric in a and b. We thus conclude that (13.5.13) is integrable if

$$\frac{\partial F^k{}_a(u, v)}{\partial u^l} F^l{}_b(u, v) - \frac{\partial F^k{}_a(u, v)}{\partial v^b} = \frac{\partial F^k{}_b(u, v)}{\partial u^l} F^l{}_a(u, v) - \frac{\partial F^k{}_b(u, v)}{\partial v^a} \tag{13.5.17}$$

for all u, v.

In order to prove that (13.5.17) is indeed satisfied, we return to the Killing vector condition (13.5.6). Multiplying with $\bar{g}^{il}$, we have

$$\frac{\partial \xi^l}{\partial v^a} = -\bar{g}^{il} \frac{\partial \xi^m}{\partial u^i} g_{ma} - \bar{g}^{il} \xi^k \frac{\partial g_{ia}}{\partial u^k}$$

Also, multiplying (13.5.5) with $\bar{g}^{il} \cdot \bar{g}^{jm}$ gives

$$\begin{aligned} \bar{g}^{il} \frac{\partial \xi^m}{\partial u^i} + \bar{g}^{jm} \frac{\partial \xi^l}{\partial u^j} &= -\xi^k \bar{g}^{il} \bar{g}^{jm} \frac{\partial g_{ij}}{\partial u^k} \\ &= \xi^k \frac{\partial \bar{g}^{lm}}{\partial u^k} \end{aligned}$$

so

$$\frac{\partial \xi^l}{\partial v^a} = \bar{g}^{jm} \frac{\partial \xi^l}{\partial u^j} g_{ma} - \xi^k \frac{\partial \bar{g}^{lm}}{\partial u^k} g_{ma} - \xi^k \bar{g}^{lm} \frac{\partial g_{ma}}{\partial u^k}$$

Recalling (13.5.14), we can write this as

$$\frac{\partial \xi^l}{\partial v^a} = F^j{}_a \frac{\partial \xi^l}{\partial u^j} - \xi^k \frac{\partial F^l{}_a}{\partial u^k} \tag{13.5.18}$$

Now differentiate with respect to v^b; this gives

$$\frac{\partial^2 \xi^l}{\partial v^b \partial v^a} = F^j{}_a \frac{\partial}{\partial u^j} \left(\frac{\partial \xi^l}{\partial v^b} \right) + \frac{\partial F^j{}_a}{\partial v^b} \frac{\partial \xi^l}{\partial u^j} - \frac{\partial \xi^k}{\partial v^b} \frac{\partial F^l{}_a}{\partial u^k} - \xi^k \frac{\partial^2 F^l{}_a}{\partial v^b \partial u^k}$$

or, using (13.5.18) on the right-hand side,

$$\begin{aligned} \frac{\partial^2 \xi^l}{\partial v^b \partial v^a} = {} & F^j{}_a F^i{}_b \frac{\partial^2 \xi^l}{\partial u^j \partial u^i} + F^j{}_a \frac{\partial F^i{}_b}{\partial u^j} \frac{\partial \xi^l}{\partial u^i} - F^j{}_a \frac{\partial F^l{}_b}{\partial u^k} \frac{\partial \xi^k}{\partial u^j} - F^j{}_a \frac{\partial^2 F^l{}_b}{\partial u^k \partial u^j} \xi^k \\ & + \frac{\partial F^j{}_a}{\partial v^b} \frac{\partial \xi^l}{\partial u^j} - F^i{}_b \frac{\partial F^l{}_a}{\partial u^k} \frac{\partial \xi^k}{\partial u^i} + \frac{\partial F^k{}_b}{\partial u^i} \frac{\partial F^l{}_a}{\partial u^k} \xi^i - \frac{\partial^2 F^l{}_a}{\partial v^b \partial u^k} \xi^k \end{aligned}$$

But this must be symmetric between a and b, so

$$0 = \left\{ F^j{}_a \frac{\partial F^i{}_b}{\partial u^j} - F^j{}_b \frac{\partial F^i{}_a}{\partial u^j} + \frac{\partial F^i{}_a}{\partial v^b} - \frac{\partial F^i{}_b}{\partial v^a} \right\} \frac{\partial \xi^l}{\partial u^i}$$
$$+ \left\{ -F^j{}_a \frac{\partial^2 F^l{}_b}{\partial u^k \partial u^j} + F^j{}_b \frac{\partial^2 F^l{}_a}{\partial u^k \partial u^j} + \frac{\partial F^i{}_b}{\partial u^k} \frac{\partial F^l{}_a}{\partial u^i} - \frac{\partial F^i{}_a}{\partial u^k} \frac{\partial F^l{}_b}{\partial u^i} \right.$$
$$\left. - \frac{\partial^2 F^l{}_a}{\partial v^b \partial u^k} + \frac{\partial^2 F^l{}_b}{\partial v^a \partial u^k} \right\} \xi^k \tag{13.5.19}$$

We have already remarked that our assumption that there are $M(M+1)/2$ independent Killing vectors allows us at any given point to find Killing vectors for which ξ^k vanishes and for which $\xi_{k;j} = g_{kl}\, \partial \xi^l / \partial u^i$ is an arbitrary antisymmetric matrix. In particular, we can at any given point choose ξ^i so that

$$\xi^k = 0$$
$$\xi_{k;i} = g_{kl} \frac{\partial \xi^l}{\partial x^i} = \delta_{km}\delta_{in} - \delta_{kn}\delta_{im}$$

Hence multiplying (13.4.19) with g_{kl} and setting $k = n \neq m$, we find that

$$F^j{}_a \frac{\partial F^m{}_b}{\partial u^j} - F^j{}_b \frac{\partial F^m{}_a}{\partial u^j} = \frac{\partial F^m{}_b}{\partial v^a} - \frac{\partial F^m{}_a}{\partial v^b}$$

which is just the desired relation (13.5.17). The coefficient of ξ^i in (13.5.19) must also vanish, but we do not need this information here.

To return now to the main line of our proof: Having proved (13.5.17), we know that (13.5.13) is integrable, so we can construct the coordinates u'^i, v'^a defined by (13.5.10) and (13.5.11), in which the metric components g'_{ia} vanish. Let us do so, and drop the primes, so that now

$$g_{ia} = 0 \tag{13.5.20}$$

The Killing vector conditions (13.5.6), (13.5.7) now read

$$0 = \frac{\partial \xi^k}{\partial v^a} g_{ik} \tag{13.5.21}$$

$$0 = \xi^k \frac{\partial g_{ab}}{\partial u^k} \tag{13.5.22}$$

Since g_{ik} is nonsingular, it follows from (13.5.21) that

$$\frac{\partial \xi^k}{\partial v^a} = 0 \tag{13.5.23}$$

Also, we have noted that at each point we can find Killing vectors for which ζ^k takes any arbitrary value, so the coefficient of ξ^k in (13.5.22) must vanish:

$$\frac{\partial g_{ab}}{\partial u^k} = 0 \tag{13.5.24}$$

It only remains to show that $g_{ij}(u, v)$ is v-independent, except for a possible factor $f(v)$. We use the fact that for any fixed v_0 there are $M(M+1)/2$ independent Killing vectors of $g_{ij}(u, v_0)$ that, according to (13.5.23), are also Killing vectors of $g_{ij}(u, v)$ for any v. Each one of these Killing vectors $\xi^i(u)$ will then satisfy (13.5.5) at $v = v_0$ and for general v:

$$0 = \frac{\partial \xi^k(u)}{\partial u^i} g_{kj}(u, v_0) + \frac{\partial \xi^k(u)}{\partial u^j} g_{ki}(u, v_0) + \xi^k(u) \frac{\partial g_{ij}(u, v_0)}{\partial u^k}$$

$$0 = \frac{\partial \xi^k(u)}{\partial u^i} g_{kj}(u, v) + \frac{\partial \xi^k(u)}{\partial u^j} g_{ki}(u, v) + \xi^k(u) \frac{\partial g_{ij}(u, v)}{\partial u^k}$$

We can interpret these two equations as saying that $g_{ij}(u, v)$ is a maximally form-invariant tensor [in the sense of Eq. (13.4.3)] in the maximally symmetric space with metric $g_{ij}(u, v_0)$. It follows then, according to Eqs. (13.4.10) and (13.4.11), that the tensor $g_{ij}(u, v)$ is proportional to the metric $g_{ij}(u, v_0)$, with a u-independent coefficient:

$$g_{ij}(u, v) = f(v, v_0) g_{ij}(u, v_0)$$

The valve of v_0 can be fixed in any way we like, so we can suppress the label v_0, and write this as

$$g_{ij}(u, v) = f(v) \tilde{g}_{ij}(u) \tag{13.5.25}$$

with

$$f(v) \equiv f(v, v_0) \qquad \tilde{g}_{ij}(u) \equiv g_{ij}(u, v_0) \tag{13.5.26}$$

Putting together (13.5.20), (13.5.24), and (13.5.25) now shows that the metric $g_{\mu\nu}(u, v)$ does have the form given by (13.5.4), and (13.5.26) and (13.5.5) with $v = v_0$ show that $\tilde{g}_{ij}(u)$ is a maximally symmetric metric, as was to be proved.

This theorem could also have been proved under the apparently weaker assumption, that the whole space can be decomposed into subspaces that are isotropic about every point. This assumption means that any point u_0, v we can find Killing vectors of the whole space with $\xi^a \equiv 0$, for which ξ^i vanishes at u_0, v, and for which $\xi_{i;k}$ at u_0, v is an arbitrary antisymmetric matrix. In particular, we can find $M(M-1)/2$ Killing vectors $\xi^{(lm)}(u, v; u_0)$ with

$$\xi^{a(lm)}(u, v; u_0) = 0$$

$$\xi^{i(lm)}(u, v; u_0) = -\xi^{i(ml)}(u, v; u_0)$$

for which

$$\xi^{(lm)}_{i;j}(u_0, v; u_0) \equiv g_{ik}(u_0, v)\left(\frac{\partial \xi^{k(lm)}(u, v; u_0)}{\partial u^j}\right)_{u=u_0}$$

$$= \delta_i{}^l \delta_j{}^m - \delta_i{}^m \delta_j{}^l$$

We can then define

$$\xi^{\mu(l)}(u, v; u_0) \equiv \frac{\partial}{\partial u_0{}^m} \xi^{\mu(lm)}(u, v; u_0)$$

and the arguments of Section 13.1 show that these are Killing vectors *of the whole space*, with

$$\xi^{a(l)}(u, v; u_0) = 0$$

and with

$$\xi^{i(l)}(u_0, v; u_0) = -\frac{1}{(N-1)} \bar{g}^{il}(u_0, v)$$

The existence of the $M(M+1)/2$ independent Killing vectors $\xi^{\mu(lm)}$ and $\xi^{\mu(l)}$ shows that the space does have maximally symmetric subspaces after all.

In all cases of practical importance, the maximally symmetric subspaces are *spaces*, as opposed to space-times, so all eigenvalues of the submatrix g_{ij} are positive. In this case, we can use (13.3.23), (13.3.24), or (13.3.25) to evaluate $\tilde{g}_{ij}\, du^i\, du^j$, and (13.5.3) then gives

$$-d\tau^2 = g_{ab}(v)\, dv^a\, dv^b + f(v)\left\{d\mathbf{u}^2 + \frac{k(\mathbf{u}\cdot d\mathbf{u})^2}{1 - k\mathbf{u}^2}\right\} \tag{13.5.27}$$

where $f(v)$ is positive and

$$k = \begin{cases} +1 & \text{if max. sym. subsp. has } K > 0 \\ -1 & \text{if max. sym. subsp. has } K < 0 \\ 0 & \text{if max. sym. subsp. has } K = 0 \end{cases} \tag{13.5.28}$$

[We have absorbed the curvature constant $|K|^{-1}$ appearing in (13.3.23) and (13.3.24) into the function $f(v)$.] Let us now use this formula to treat the special cases listed in Table 13.1:

(A) **Spherically Symmetric Space.** Suppose that the dimensionality of the whole space is $N = 3$, that all eigenvalues of its metric are positive, and that it has maximally symmetric two-dimensional subspaces with positive curvature. Then there is one v-coordinate, which we can call r, and two u-coordinates, which we can replace with angles θ, φ defined by

$$u^1 = \sin\theta\cos\varphi \qquad u^2 = \sin\theta\sin\varphi \tag{13.5.29}$$

Eq. (13.5.27), with $k = 1$, then gives

$$ds^2 = g(r)\,dr^2 + f(r)\,\{d\theta^2 + \sin^2\theta\,d\varphi^2\} \tag{13.5.30}$$

with $f(r)$ and $g(r)$ positive functions of r.

(B) **Spherically Symmetric Space-Time.** Suppose that the dimensionality of the whole space-time is $N = 4$, that three of the eigenvalues of its metric are positive and one is negative, and that it has maximally symmetric two-dimensional subspaces whose metric has positive eigenvalues and positive curvature. Then there are two v-coordinates, which we can call r and t, and two u-coordinates, which can be replaced with θ and φ as in (13.5.29). Eq. (13.5.27), with $k = 1$, then gives

$$\begin{aligned} -d\tau^2 = {} & g_{tt}(r, t)\,dt^2 + 2g_{rt}(r, t)\,dr\,dt + g_{rr}(r, t)\,dr^2 \\ & + f(r, t)\,\{d\theta^2 + \sin^2\theta\,d\varphi^2\} \end{aligned} \tag{13.5.31}$$

where $f(r, t)$ is a positive function and $g_{ij}(r, t)$ is a 2×2 matrix with one positive and one negative eigenvalue.

(C) **Spherically Symmetric Homogeneous Space-Time.** Suppose that the dimensionality of the whole space-time is $N = 4$, that three of the eigenvalues of its metric are positive and one is negative, and that it has maximally symmetric *three*-dimensional subspaces whose metric has positive eigenvalues and arbitrary curvature. Then there is one v-coordinate and three u-coordinates, and (13.5.27) gives

$$-d\tau^2 = g(v)\,dv^2 + f(v)\left\{d\mathbf{u}^2 + \frac{k(\mathbf{u}\cdot d\mathbf{u})^2}{1 - k\mathbf{u}^2}\right\}$$

where $f(v)$ is a positive function and $g(v)$ is a negative function of v. It is very convenient to define new coordinates t, v, θ, φ by

$$\int(-g(v))^{1/2}\,dv \equiv t$$

$$\begin{aligned} u^1 &\equiv r\sin\theta\cos\varphi \\ u^2 &\equiv r\sin\theta\sin\varphi \\ u^3 &\equiv r\cos\theta \end{aligned}$$

We then have

$$d\tau^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2\right\} \tag{13.5.32}$$

where $R(t) \equiv \sqrt{f(v)}$.

The first two examples show how it is possible to capture the essence of spherical symmetry by giving a qualitative description of a space or space-time in terms of dimensionalities, signs of eigenvalues and curvatures, and the maximal

symmetry of its subspaces. The metrics (13.5.30) and (13.5.31) are just what we would have expected on more elementary grounds; indeed, (13.5.31) was our starting point in Section 11.7.

On the other hand, our third example leads to a result that could not easily have been anticipated. It is true that Eq. (13.5.32) was already derived in Section 11.9 as the metric inside a spherically symmetric collapsing star of uniform density and zero pressure. The beautiful new thing we have learned in this chapter is that this metric can be derived solely from the assumption of homogeneity and isotropy, with no use of the Einstein field equations.

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